PREPRINT UCRL- 81637

Lawrence Livermore Laboratory

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December 20, 1978

1979 IEEE International Conference on Acoustics, Speech, and Signal Processing, Washington, DC, April 2-4, 1979.

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ABSTRACT

The relative advantages of 2-D recursive digital filters over their nonrecursive counterparts are discussed. A design example illustrates the ability of 2-D recursive filters to yield excellent responses with far fewer coefficients than nonrecursive filters require. This difficulty is seen to be partially overcome by using nonrecursive filters with very efficient implementations.

INTRODUCTION

In recent years, there has been a rapid increase in the number of applications requiring the digital processing of two-dimensional (2-D) signals. Typical areas of application have included; 1) satellite-borne remote photography for the monitoring of environmental effects, earth resources, and urban land use; 2) processing of geological and seismological data in the exploration for oil and natural gas; 3) 3-D imaging of the brain using multiple 2-D projection techniques; and 4) the processing of medical and industrial radiographs.

In most 2-D signal processing applications, the goal is to somehow extract some desired information from a 2-D data array by performing appropriate operations on that data. Perhaps the most common method of extracting the desired information is via 2-D digital filtering. As in 1-D, there are two filtering implementations in the 2-D case: nonrecursive and recursive. Nonrecursive 2-D digital filters can be characterized by the difference equation

$$y(m,n) = \sum_{k} \sum_{\ell} a(k,\ell)\ x\ (m-k,n-\ell) \quad (1)$$

*Work performed under the auspices of the U. S. Department of Energy by the Lawrence Livermore Laboratory under contract number W-7405-ENG-48 and supported in part by NSF Grant ENG-78-04240."

where y is the output array, a is the filter array (finite extent), and x is the input array. The more general recursive filters are characterized by the equation

$$\sum_{k} \sum_{\ell} b(k,\ell) y(m-k,n-\ell) = \sum_{k} \sum_{\ell} a(k,\ell) x(m-k,n-\ell) \quad (2)$$

From a transfer function viewpoint, nonrecursive filters have only zeros, i.e., the transfer function is simply a polynomial in the two unit shift variables z_1^{-1} and z_2^{-1}. Recursive filters, on the other hand, have transfer function relationships consisting of a ratio of polynomials, and, therefore, possess both poles and zeros. Historically, the arguments for using 1-D recursive filters have been that they yield more efficient implementations in terms of both storage and computation; that is, the design flexibility inherent in allowing the transfer function to have both poles and zeros results in better filter characteristics, for a fixed total number of filter coefficients, than can be achieved with nonrecursive filters. The principal purpose of this paper is to present results that indicate that 2-D recursive filters apparently enjoy this same advantage over their nonrecursive counterparts.

FILTER DESIGN COMPARISONS

There have been many procedures proposed for the design of 2-D nonrecursive digital filters, including those found in (1) - (6). Several of these techniques are direct extensions of 1-D filter design procedures: the windowing approaches discussed in (1) and (7) and the frequency sampling design procedure discussed in (2) are examples. Other techniques include various 2-D linear programming problems (3) and transformations of 1-D filters to 2-D (4), (5).

Several 2-D recursive filter design procedures have similarly been proposed (7) - (10); here one must ensure that the filter be stable in addition to approximating the desired specification. Most of these design procedures have assumed a quarter-plane support for the filter. Murray (11) has shown, however, that such a constraint leads to some severe restrictions on the resulting frequency response of the filter. A design procedure for the more general class of half-plane filters (which are capable of realizing an arbitrary magnitude specification) has recently been proposed in (12) and elaborated upon in (13).

A complete study of the comparative capabilities of the numerous 2-D nonrecursive and recursive filter design procedures is a formidable task and beyond the scope of this presentation. Instead, we will restrict our attention to a comparison of the half-plane recursive filter design results of (13) and the nonrecursive filter design results, via the generalized McClellan transformation of (5). Although the example discussed below contains just one nonrecursive and one recursive design procedure, our experience with other examples and other procedures indicates that it is fairly representative of the relative capabilities of recursive and nonrecursive filters in two dimensions.

The desired specification for the comparative designs was taken to be

$$S(u,v) = \begin{cases} 1 & |\Theta| \leq \pi/4 \text{ or } |\Theta| \geq 3\pi/4 \\ .02 & \text{otherwise} \end{cases}$$

where $\Theta = \arctan \frac{v}{u}$.

This corresponds to the 90° fan filter specification illustrated by Figure 1. Several half-plane recursive filters, of various orders, were designed to approximate this specification. The design procedure, described in detail in (13), utilizes a nonlinear optimization algorithm

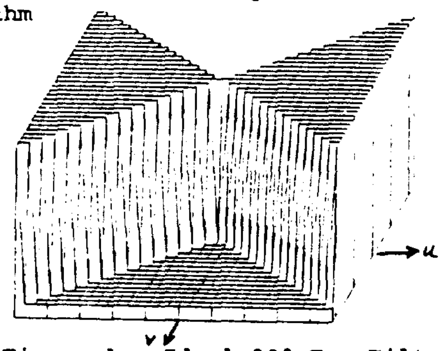


Figure 1. Ideal 90° Fan Filter Specification

with the stability of the resulting (locally) optimum filter ensured via a spectral factorization stability test (14). A weighted least squares minimization was performed in the spatial frequency domain (using a 32 x 32 DFT to evaluate the approximation errors); i.e., the weighted ℓ_2 error

$$\| W(u_i,v_j) \; [s(u_i,v_j) - |H(u_i,v_j)|^2] \| \quad (3)$$

was minimized subject to the stability constraint. The weighting function in this example was chosen to be 5 in the passband, 0 in the transition band (any point within 3 grid points of the passband), and 1 in the stopband. The term $|H(u_i,v_j)|$ is the magnitude response of the recursive filter, and the squaring operation results from the assumption of a zero-phase implementation (i.e., 2 recursions of the same filter are used in opposing directions to achieve exactly zero phase).

The optimal 3 x 3 half-plane denominator with 4 x 4 numerator result (a total of only 41 coefficients) at convergence was

$$\frac{\begin{array}{cccc} 0.0103 & 0.0546 & -0.0483 & -0.0077 \\ 0.0627 & 0.3232 & 0.0663 & 0.0369 \\ -0.4953 & -0.7232 & -0.4309 & 0.0365 \\ 0.4130 & 0.6390 & 0.2045 & 0.0364 \end{array}}{\begin{array}{ccccccc} 0.0012 & 0.0003 & -0.0494 & -0.0812 & -0.0741 & -0.0044 & -0.0052 \\ 0.0024 & 0.1044 & 0.0665 & 0.6778 & 0.1526 & 0.2022 & 0.0144 \\ -0.0083 & -0.0721 & -0.8787 & 0.0103 & -1.0150 & -0.2063 & 0.0024 \\ & & & 1.1825 & 0.1992 & -0.0335 & -0.0115 \end{array}}$$

A plot of the magnitude (squared) response of this filter is shown in Figure 2. Note that the resulting recursive filter yields an excellent approximation to the spatial frequency specification of Figure 1.

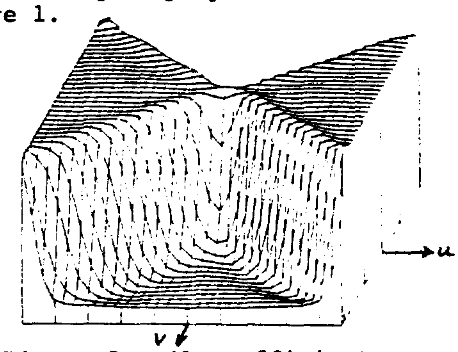


Figure 2. 41-coefficient recursive filter response.

For the nonrecursive filter design, we implemented the generalized McClellan transformation approach of (5). This technique transforms a 1-D zero-phase nonrecursive filter into a 2-D zero-phase nonrecursive filter via the substitution

$$\cos \omega = \sum_{p=0}^{P} \sum_{q=0}^{Q} t(p,q) \cos p\omega_1 \cos q\omega_2 \; . \quad (4)$$

We chose this design method because it gives good design results with only limited computation required and because it leads, as will be discussed in the following section, to a very efficient implementation which stands the best chance of comparing favorably with the more powerful recursive filters.

For the 90° fan design problem at hand, a first-order mapping defined by $t(0,0) = t(1,1) = 0$, $t(0,1) = -t(1,0) = 0.5$ suffices (5). Several 1-D prototype filters were designed via the Remez exchange algorithm of (15), resulting (after transformation) in the 15 x 15, 19 x 19, 23 x 23, and 27 x 27 fan filters shown in Figure 3. Note that the nonrecursive filter performances cannot compete with the 41-coefficient recursive filter of Figure 2 unless quite a large filter support is chosen (say, at least 27 x 27 in this case). A quantitative comparison is difficult due to the different error norms used in the designs, but visual inspection gives a good approximation to the trade-offs involved. Other design comparisons (for low-pass and other fan filters) indicate to us that this is a fairly representative example. We have observed that nonrecursive filters with 400-1000 coefficients are needed to compete with recursive filters with 40-100 coefficients; i.e., about a factor of 10 more coefficients are required.

IMPLEMENTATION ISSUES

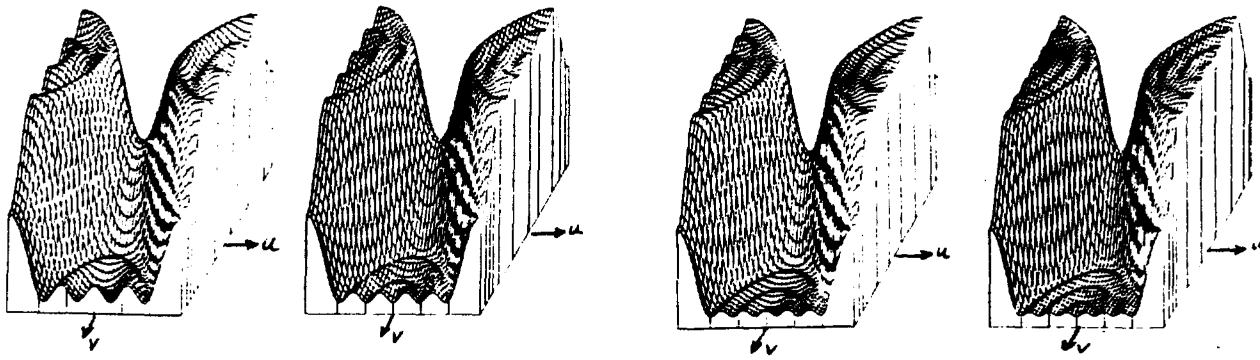
The overriding reason for using recursive filters to filter imagery is, as in 1-D, that they possess more efficient (in terms of computation and storage) implementations than nonrecursive filters. This is because, as the above example illustrated, fewer filter coefficients are generally required.

For simplicity, let us designate the total number of recursive filter coefficient N_R; e.g., for the 3 x 3 half-plane with 4 x 4 numerator, $N_R = 41$. It is readily verified that the recursive filtering requires (N_R-1) multiplies per pixel and (N_R-1) adds per pixel. Furthermore, the I/O is very straightforward because, due to the small filter sizes, only a few rows of the image need be present in memory at a time. This will be trivial in all cases except when the image is extremely large or the computer memory is extremely limited -- in those cases, other relatively straightforward schemes are available that still require modest I/O of data.

The implementation of nonrecursive filters is an entirely different matter. Several schemes have been used in the past, including direct convolution, FFT approaches (16) - (18), and sectioned FFT approaches (19). Furthermore, the filters designed via the generalized McClellan transformation possess their own extremely efficient implementation (20).

Direct convolution is an efficient implementation of 2-D nonrecursive filters only when the filter support is very small. For an $N_1 \times N_2$ filter, N_1N_2 multiplies and (N_1N_2-1) adds per pixel are necessary. As discussed in (20), this is slower than other techniques when filter supports exceed approximately 10 x 10. With zero-phase filters, the centrosymmetry of the coefficient array reduces the number of multiplies by a factor of 2.

FFT implementations can be very efficient, particularly for large filter sizes. This is because the FFT computations depend on the image size rather than the filter size (at least when the image is much larger than the filter). The 2-D FFT implementation requires, for



a) 15 x 15 filter response.

b) 19 x 19 filter response.

c) 23 x 23 filter response.

d) 27 x 27 filter response.

Figure 3. Various nonrecursive 2-D filter responses designed via the generalized McClellan transformation.

an $M \times M$ image (M a power of 2), $4M^2 \log_2 M + M^2$ real multiplies and $6M^2 \log_2 M + M^2$ adds. It therefore requires approximately $4 \log_2 M+1$ multiplies and $6 \log_2 M+1$ adds per pixel. The I/O, however, can be quite costly for FFT implementations when the computer memory is significantly smaller than the image to be filtered (16); in such cases, sectioned approaches can be used which generally decrease I/O requirements at the cost of additional CPU computation (19).

The generalized McClellan filters discussed above have a very efficient special implementation which arises because of the 1-D nature of the filter prior to transformation. For a 2N+1 length 1-D filter and a PxQ transformation, it can be shown (20) that the resulting (2NP+1)x(2NQ+1) 2-D filter is implementable with approximately $(P+1)^2N+N+1$ multiplies per pixel and $(2P+1)^2N+N$ adds per pixel. As with the FFT implementation, I/O problems arise for large images with this technique also; sectioned techniques can again be employed in this case with a modest increase in CPU computation (multiplies and adds) resulting.

For the fan filter example above, the various implementation possibilities can be summarized as in Table 1. It is apparent that the "single-recursion" half-plane filter possesses a superior implementation, whereas the FFT, generalized McClellan implementation, and the zero-phase recursive filter are approximately equal. For computers with limited available primary memory, the I/O costs of the nonrecursive implementations may prove excessive also, again making the recursive implementation attractive. One must, of course, evaluate CPU vs. I/O trade-offs and programming considerations before making any such judgment for a given computer architecture.

		# Multiplies Per Pixel	# Adds Per Pixel	Total # Floating Point Operations Per Pixel
41-coefficient recursive filter:	single recursion	40	40	80
	zero-phase (2 recursions)	80	80	160
27 x 27 nonrecursive filter:	direct convolution	729	728	1457
	FFT	41	61	102
	Special McClellan Filter Implementation	66	131	197
	Utilizing fact that transformation coeffs. are power of 2 for this example	0	131	131

Table 1. CPU comparisons of various filter implementations (assumed 1024 x 1024 image).

CONCLUSIONS

Some of the issues involved in comparing nonrecursive and recursive 2-D digital filters have been discussed. Recursive filters were seen to approximate a given specification with much smaller filter supports than nonrecursive filters. The attendant advantage in filter implementation is lessened, however, by the fast implementation schemes enjoyed by 2-D nonrecursive filter forms.

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